The generalized 3-connectivity of Cartesian product graphs*

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Abstract

The generalized connectivity of a graph, which was introduced recently by Chartrand et al., is a generalization of the concept of vertex connectivity. Let S be a nonempty set of vertices of G, a collection $\{T_1, T_2, \ldots, T_r\}$ of trees in G is said to be internally disjoint trees connecting S if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of distinct integers i, j, where $1 \leq i, j \leq r$. For an integer k with $2 \leq k \leq n$, the k-connectivity $\kappa_k(G)$ of G is the greatest positive integer r for which G contains at least r internally disjoint trees connecting S for any set S of k vertices of G. Obviously, $\kappa_2(G) = \kappa(G)$ is the connectivity of G. Sabidussi showed that $\kappa(G\square H) \geq \kappa(G) + \kappa(H)$ for any two connected graphs G and H. In this paper, we first study the 3-connectivity of the Cartesian product of a graph G and a tree T, and show that (i) if $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G\square T) \geq \kappa_3(G)$; (ii) if $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G\square T) \geq \kappa_3(G) + 1$. Furthermore, for any two connected graphs G and H with $\kappa_3(G) \geq \kappa_3(H)$, if $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + \kappa_3(H)$; if $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G\square H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Our result could be seen as a generalization of Sabidussi's result. Moreover, all the bounds are sharp.

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1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretic notations and terminology not described here. Let G be a graph, the connectivity $\kappa(G)$ of a graph G is defined as $min\{|S|\,|\, S \subseteq V(G)\ and\ G-S\ is\ disconnected\ or\ trivial\}$. Whitney [12] showed an equivalent definition of the connectivity of a graph. For each pair of vertices x, y of G, let $\kappa(x, y)$ denote the maximum number of internally disjoint paths connecting x and y in G. Then the connectivity $\kappa(G)$ of G is $min\{\kappa(x, y)\,|\, x, y\ are\ distinct\ vertices\ of G\}$.

The Cartesian product of graphs is an important method to construct a bigger graph, and plays a key role in design and analysis of networks. In the past several decades, many authors have studied the (edge) connectivity of the Cartesian product graphs. For example, Sabidussi derived the following result about the connectivity of Cartesian product graphs.

Theorem 1.1. [10] *Let G and H be two connected graphs. Then $\kappa(G\square H) \geq \kappa(G) + \kappa(H)$.*

More information about the (edge) connectivity of the Cartesian product graphs can be found in [3, 4, 5, 10, 13].

The generalized connectivity of a graph G, which was introduced recently by Chartrand et al. in [2], is a natural and nice generalization of the concept of vertex connectivity. A tree T is called an *S-tree* (*$\{u_1, u_2, \ldots, u_k\}$-tree*) if $S \subseteq V(T)$, where $S = \{u_1, u_2, \ldots, u_k\} \in V(G)$. A family of trees $T_1, T_2, \ldots, T_r$ are *internally disjoint S-trees* if $E(T_i) \cap E(T_j) = \emptyset$ and $V(T_i) \cap V(T_j) = S$ for any pair of integers i and j, where $1 \leq i < j \leq r$. We use $\kappa(S)$ to denote the greatest number of internally disjoint S-trees. For an integer k with $2 \leq k \leq n$, the *k-connectivity* $\kappa_k(G)$ of G is defined as $min\{\kappa(S)\,|\, S \in V(G)\ and\ |S| = k\}$. Clearly, when $|S| = 2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G) = \kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G) = 1$. For any graph G, clearly, $\kappa(G) \geq 1$ if and only if $\kappa_3(G) \geq 1$.

In addition to being a natural combinatorial measure, the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree for connecting a set of vertices is usually called a Steiner tree, and popularly used

in the physical design of VLSI, see [11]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G.

In [7], Li and Li investigated the complexity of determining the generalized connectivity and derived that for any fixed integer $k \geq 2$, given a graph G and a subset S of $V(G)$, deciding whether there are k internally disjoint trees connecting S, namely deciding whether $\kappa(S) \geq k$, is NP-complete.

Chartrand et al. [2] got the following result for complete graphs.

Theorem 1.2. [2] *For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil k/2 \rceil$.*

Okamoto and Zhang [9] investigated the generalized connectivity for regular complete bipartite graphs $K_{a,a}$. Recently, Li et al. [6] got the following result for general complete bipartite graphs.

Theorem 1.3. [6] *Given any two positive integers a and b, let $K_{a,b}$ denote a complete bipartite graph with a bipartition of sizes a and b, respectively. Then we have the following results: if $k > b - a + 2$ and $a - b + k$ is odd then*

$$\kappa_k(K_{a,b}) = \frac{a+b-k+1}{2} + \lfloor \frac{(a-b+k-1)(b-a+k-1)}{4(k-1)} \rfloor;$$

if $k > b - a + 2$ and $a - b + k$ is even then

$$\kappa_k(K_{a,b}) = \frac{a+b-k}{2} + \lfloor \frac{(a-b+k)(b-a+k)}{4(k-1)} \rfloor;$$

and if $k \leq b - a + 2$ then

$$\kappa_k(K_{a,b}) = a.$$

Li et al. [8] got the following upper bounds of $\kappa_3(G)$ for general graphs.

Theorem 1.4. [8] *Let G be a connected graph with at least three vertices. If G has two adjacent vertices with minimum degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Theorem 1.5. [8] *Let G be a connected graph with n vertices. Then, $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

In this paper, we study the 3-connectivity of Cartesian product graphs. The paper is organized as follows. In Section 2, we recall the definition and properties of

Cartesian product graphs, and give some basic results about the internally disjoint S-trees. In Sections 3 and 4, we study the 3-connectivity of the Cartesian product of a graph G and a tree T, and show that (i) if $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G\Box T) \geq \kappa_3(G)$; (ii) if $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G\Box T) \geq \kappa_3(G) + 1$. Moreover, the bounds are sharp. As a consequence, we get that $\kappa_3(Q_n) = n - 1$, where Q_n is the n-hypercube. In Section 5, we study the 3-connectivity of the Cartesian product of two connected graphs G and H, and show that for any two connected graphs G and H with $\kappa_3(G) \geq \kappa_3(H)$, if $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G\Box H) \geq \kappa_3(G) + \kappa_3(H)$; if $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G\Box H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Moreover, all the bounds are sharp. Our result could be seen as a generalization of Theorem 1.1.

2 Some basic results

We use P_n to denote a path with n vertices. A path P is called a *u-v path*, denoted by $P_{u,v}$, if u and v are the endpoints of P.

Recall that the *Cartesian product* (also called the *square product*) of two graphs G and H, written as $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G\Box H \cong H\Box G$. The edge $(u, v)(u', v')$ is called *one-type edge* if $(u, u') \in E(G)$ and $v = v'$; similarly, the $(u, v)(u', v')$ is called *two-type edge* if $u = u'$ and $(v, v') \in E(H)$.

Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. We use $G(u_j, v_i)$ to denote the subgraph of $G\Box H$ induced by the set $\{(u_j, v_i) \,|\, 1 \leq j \leq n\}$. Similarly, we use $H(u_j, v_i)$ to denote the subgraph of $G\Box H$ induced by the set $\{(u_j, v_i) \,|\, 1 \leq i \leq m\}$. It is easy to see $G(u_{j_1}, v_i) = G(u_{j_2}, v_i)$ for different u_{j_1} and u_{j_2} of G. Thus, we can replace $G(u_j, v_i)$ by $G(v_i)$ for simplicity. Similarly, we can replace $H(u_j, v_i)$ by $H(u_j)$. For $x = (u, v)$, we refer to (u, v') and (u', v) as *the vertices corresponding to* x in $G(v')$ $(= G(u, v'))$ and $H(u')$ $(= H(u', v))$, respectively. Similarly, we can define the path and tree corresponding to some path and tree, respectively.

Imrich and Klavžar gave the following result in [4].

Proposition 2.1. [4] *The Cartesian product of two graphs G and H is connected if and only if both graphs G and H are connected.*

By Proposition 2.1, we only consider the generalized connectivity $\kappa_3(G)$ of the Cartesian product of two connected graphs.

Proposition 2.2. [4] *The Cartesian product is associative, that is,* $(G_1\square G_2)\square G_3 \cong G_1\square(G_2\square G_3)$.

In order to show our main results, we need the following well-known result.

Theorem 2.1 (Menger's Theorem [1])**.** *Let G be a k-connected graph, and let x and y be a pair of distinct vertices in G. Then there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ in G connecting x and y.*

Let G be a connected graph, and $S = \{x_1, x_2, x_3\} \subseteq V(G)$. We first have the following observation about internally disjoint S-trees.

Observation 2.1. *Let G be a connected graph, $S = \{x_1, x_2, x_3\} \subseteq V(G)$, and T be an S-tree. Then there exists a subtree T' of T such that T' is also an S-tree such that $1 \leq d_{T'}(x_i) \leq 2$, $|\{x_i \,|\, d_{T'}(x_i) = 1\}| \geq 2$ and $\{x \,|\, d_{T'}(x) = 1\} \subseteq S$. Moreover, if $|\{x_i \,|\, d_{T'}(x_i) = 1\}| = 3$, then all the vertices of $V(T') \setminus \{x_1, x_2, x_3\}$ have degree 2 except for one vertex, say x with $d_{T'}(x) = 3$; if there exists one vertex of S, say x_1, has degree 2 in T', then T' is an x_2-x_3 path.*

Proof. It is easy to check that this observation holds by deleting vertices and edges of T. $\square$

Remark 2.1. (i) *Since the path between any two distinct vertices is unique in T, the tree T' obtained from T is unique in Observation* 2.1*. Such a tree is called a minimal S-tree (or minimal $\{x_1, x_2, x_3\}$-tree).*

(ii) *Let $S = \{x, y, z\} \subseteq V(G)$. Throughout this paper, we can assume that each S-tree is a minimal S-tree.*

Lemma 2.1. *Let G be a graph with $\kappa_3(G) = k \geq 2$, $S = \{x, y, z\} \subseteq V(G)$. Then, we have the following result.*

(i) *If $G[S]$ is a clique, then there exist k internally disjoint S-trees $T_1, T_2, \ldots, T_k$, such that $E(T_i) \cap E(G[S]) = \emptyset$ for $1 \leq i \leq k-2$.*

(ii) *If $G[S]$ is not a clique, then there exist k internally disjoint S-trees $T_1, T_2, \ldots, T_k$, such that $E(T_i) \cap E(G[S]) = \emptyset$ for $1 \leq i \leq k-1$.*

Proof. We first prove (i). Clearly, by the definition of S-trees, we know $|\{T_i \,|\, E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 3$. Let $\{T_1, T_2, \ldots, T_k\}$ be k internally disjoint S-trees. If $|\{T_i \,|\, E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 2$, we are done by exchanging subscript. Thus, suppose $|\{T_i \,|\, E(T_i) \cap E(G[S]) \neq \emptyset\}| = 3$. Without loss of generality, we assume $E(T_i) \cap E(G[S]) \neq \emptyset$, where $i = k-2, k-1, k$. It is easy to check that T_{k-2}, T_{k-1}, T_k

must have the structures as shown in Figures 1a and 1b. But, for these two cases, we can obtain T'_{k-2}, T'_{k-1}, T'_k from T_{k-2}, T_{k-1}, T_k, such that $E(T'_{k-2}) \cap \{xy, xz, yz\} = \emptyset$. See Figs. 1$c$. and 1$d$, where the tree T'_{k-2} is shown by dotted lines. Thus $T_1, T_2, \ldots, T_{k-3}, T'_{k-2}, T'_{k-1}, T'_k$ are our desired S-trees.

The proof of (ii) is similar to that of (i), and thus is omitted. $\square$

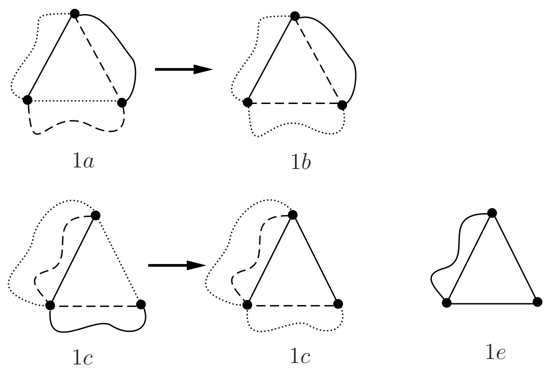

Figure 1. T'_{k-2}, T'_{k-1}, T'_k. An edge is shown by a straight line.
The edges (or paths) of a tree are shown by the same type of lines.

Remark 2.2. *Let G be a graph with $\kappa_3(G) = k \geq 2$, $S = \{x, y, z\} \subseteq V(G)$. If $|\{E(T_i) \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| \geq 2$ for any collection $\mathcal{T}$ of k internally disjoint S-trees, then $G[S]$ is a clique. Moreover, $T_{k-1} \cup T_k$ must have the structure as shown in Figure* 1e.

3 The Cartesian product of a connected graph and a path

In this section, we show the following theorem.

Theorem 3.1. *Let G be a graph and P_m be a path with m vertices. We have the following results.*

(i) If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G \square P_m) \geq \kappa_3(G)$. Moreover, the bound is sharp.

(ii) If $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G \square P_m) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp.

We shall prove Theorem 3.1 by a series of lemmas. Since the proofs of (i) and (ii) are similar, we only show (ii). Let G be a graph with $V(G) = \{u_1, u_2, \ldots, u_n\}$ such

that $1 \leq \kappa_3(G) < \kappa(G)$, $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ such that v_i and v_j are adjacent if and only if $|i - j| = 1$.

Set $\kappa_3(G) = k$ for simplicity. To prove (ii), it suffices to prove that for any $S = \{x, y, z\} \subseteq V(G \Box H)$, there exist $k+1$ internally disjoint S-trees. We proceed our proof by the following three lemmas.

Lemma 3.1. *If x, y, z belongs to the same $V(G(v_i))$, $1 \leq i \leq m$, then there exist $k+1$ internally disjoint S-trees.*

Proof. Without loss of generality, we assume $x, y, z \in V(G(v_1))$. Since $\kappa_3(G) = k$, there exist k internally disjoint S-trees $T_1, T_2, \ldots, T_k$ in $G(v_1)$. We need another S-tree T_{k+1} such that T_{k+1} and T_i are internally disjoint, where $i = 1, 2, \ldots, k$. Let x', y', z' be the vertices corresponding to x, y, z in $G(v_2)$, and T_1' be the tree corresponding to T_1 in $G(v_2)$. Therefore, The tree T_{k+1} obtained from T_1' by adding three edges xx', yy', zz' is a desired tree. $\square$

Lemma 3.2. *If exact two of x, y, z are contained in some $G(v_i)$, then there exist $k+1$ internally disjoint S-trees.*

Proof. We may assume $x, y \in V(G(v_1)), z \in V(G(v_2))$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let x', y' be the vertices corresponding to x, y in $G(v_2)$, z' be the vertex corresponding to z in $G(v_1)$. Consider the following two cases.

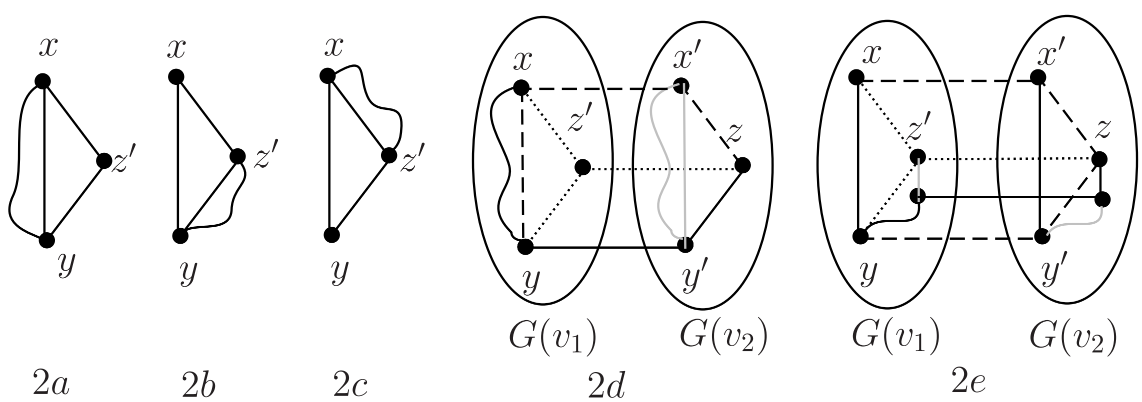

Figure 2. The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in T_i^*.

Case 1. $z' \notin \{x, y\}$.

Let $S' = \{x, y, z'\}$, and $T_1, T_2, \ldots T_k$ be k internally disjoint S'-trees in $G(v_1)$ such that $|\{T_i \,|\, E(T_i) \cap E(G(v_1)[S']\} \neq \emptyset|$ is as small as possible. We can assume that $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ for each i, where $1 \leq i \leq k-2$ by Lemma 2.1.

Subcase 1.1. $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ for each i, where $1 \leq i \leq k$.

Assume that $d_{T_i}(z') = 1$ for $1 \leq i \leq k_1$ and $d_{T_i}(z') = 2$ for $k_1 + 1 \leq i \leq k$. If $k_1 = k$, let T_i^* be the tree obtained from T_i by adding $z_i z_i'$ and $z_i' z$, and deleting z', where $1 \leq i \leq k-1$ and z_i is the only neighbor of z' in T_i, and Z_i' is the vertex corresponding to z_i in $G(v_2)$. Let $T_k^* = T_k + zz'$ and $T_{k+1}^* = T_k' + xx' + yy'$, where T_k' is the tree in $G(v_2)$ corresponding to T_k. Thus $T_1^*, T_2^*, \ldots, T_{k+1}^*$ are $k+1$ internally disjoint S-trees.

Now suppose $k_1 < k$. For $1 \leq i \leq k_1$, we construct T_i^* similar to the above procedure. For $k_1 + 1 \leq i \leq k-1$, let T_i^* be the tree obtained from T_i by adding $z_{i,1} z_{i,1}', z_{i,1}' z, z_{i,2} z_{i,2}'$ and $z_{i,2}' z$ and deleting z', where $k_1 + 1 \leq i \leq k$, $N_{T_i}(z') = \{z_{i,1}, z_{i,2}\}$, and $z_{i,1}'$ and $z_{i,2}'$ are the vertices corresponding to $z_{i,1}$ and $z_{i,2}$ in $G(v_2)$, respectively. Let $T_k^* = T_k + zz'$ and $T_{k+1}^* = T_k' + xx' + yy'$, where T_k' is the tree in $G(v_2)$ corresponding to T_k. Thus $T_1^*, T_2^*, \ldots, T_{k+1}^*$ are $k+1$ internally disjoint S-trees.

Subcase 1.2. $E(T_i) \cap E(G(v_1)[S']) \neq \emptyset$ for some i, where $i = k-1, k$.

For a tree T_i with $E(T_i) \cap E(G(v_1)[S']) = \emptyset$, we can construct $T_i{}^*$ similar to that of Subcase 1.1.

If $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$ and $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$, say $y'z \in E(T_k) \cap E(G(v_1)[S'])$. Let $T_k^* = T_k + zz'$ and $T_{k+1}^* = T_k' + xx' + yy'$, where T_k' is the tree corresponding to T_k in $G(v_2)$.

If $E(T_{k-1}) \cap E(G(v_1)[S']) \neq \emptyset$ and $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$. Then $T_{k-1} \cup T_k$ must have one of the structures as shown in Figures 2a, 2b and 2c by Remark 2.2. If T_{k-1} and T_k have the structures as shown in Figure 2a, then we can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* as shown in Figure 2d. If T_{k-1} and T_k have the structures as shown in Figure 2b, then we can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* as shown in Figure 2e. If T_{k-1} and T_k have the structures as shown in Figure 2c, then we can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* similar to those in Figure 2d.

Case 2. $z' \in \{x, y\}$.

Without loss of generality, assume $z' = y$. Since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, there exist at least $k+1$ internally disjoint x-y paths $P^1, P^2, \ldots, P^{k+1}$. Assume that y_i is the only neighbor of y in P^i, and that y_i' is the vertex corresponding to y_i in $G(v_2)$. If x and y are nonadjacent in P^i, let T_i be the tree obtained from P^i by adding $y_i y_i'$ and $y_i' z$. If x and y are adjacent in P^i, let T_i be the tree obtained from P^i by adding yz. Since G is a simple graph, there exists at most one path P^i such that x and y are adjacent on P^i. Thus $T_i, 1 \leq i \leq k+1$, are $k+1$ internally

disjoint S-trees. $\square$

Lemma 3.3. *If x, y, z are contained in distinct $G(v_i)$s, then there exist $k+1$ internally disjoint S-trees.*

Proof. We may assume that $x \in V(G(v_1)), y \in V(G(v_2)), z \in V(G(v_3))$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. Let y', z' be the vertices corresponding to y, z in $G(v_1)$, x', z'' be the vertices corresponding to x, z in $G(v_2)$ and x'', y'' be the vertices corresponding to x, y in $G(v_3)$. We consider the following three cases.

Case 1. x, y', z' are distinct vertices in $G(v_1)$

Let $S' = \{x, y', z'\}$, and $T_1, T_2, \ldots T_k$ be k internally disjoint S'-trees in $G(v_1)$ such that $|\{T_i \,|\, E(T_i) \cap E(G(v_1)[S']) \neq \emptyset\}|$ is as small as possible. We can assume that $E(T_i) \cap E(G(v_1)[S']) = \emptyset$ for each i, where $1 \leq i \leq k-2$ by Lemma 2.1. For each T_i such that $E(T_i) \cap E(G(v_1)[S']) = \emptyset$, we can obtain an S-tree T_i^* from T_i similar to that in Subcase 1.1 of Lemma 3.2.

If $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$ or $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$. Without loss of generality, we assume $E(T_{k-1}) \cap E(G(v_1)[S']) = \emptyset$. Let T_k^* be the tree obtained from T_k by adding edges $y'y, z'z''$ and $z''z$, T_{k+1}^* be the tree obtained from T_k'' by adding $x''x', x'x$ and $y''y$, where T_k'' is the tree corresponding to T_k in $G(v_3)$. Thus, T_i^*s, $1 \leq i \leq k+1$, are k internally disjoint S-tree.

Otherwise, that is, $E(T_{k-1}) \cap E(G(v_1)[S']) \neq \emptyset$ and $E(T_k) \cap E(G(v_1)[S']) \neq \emptyset$. Then T_{k-1} and T_k must have the structures as shown in Figures 3a, 3b and 3c. If T_{k-1} and T_k have the structures as shown in Figure 3a, then we can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* as shown in Figure 3d. If T_{k-1} and T_k have the structures as shown in Figure 3b, then we can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* as shown in Figure 3e. If T_{k-1} and T_k have the structures as shown in Figure 3c, then we can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* as shown in Figure 3f.

Case 2. Two of x, y', z' are the same vertex in $G(v_1)$.

If $y' = z'$, since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, it is easy to construct $k+1$ internally disjoint S-trees. See Figure 3g. The other cases ($x = y'$ or $x = z'$) can be proved with similar arguments.

Case 3. x, y', z' are the same vertex in $G(v_1)$.

Since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, it is easy to construct $k+1$ internally disjoint S-trees. See Figure 3h. $\square$

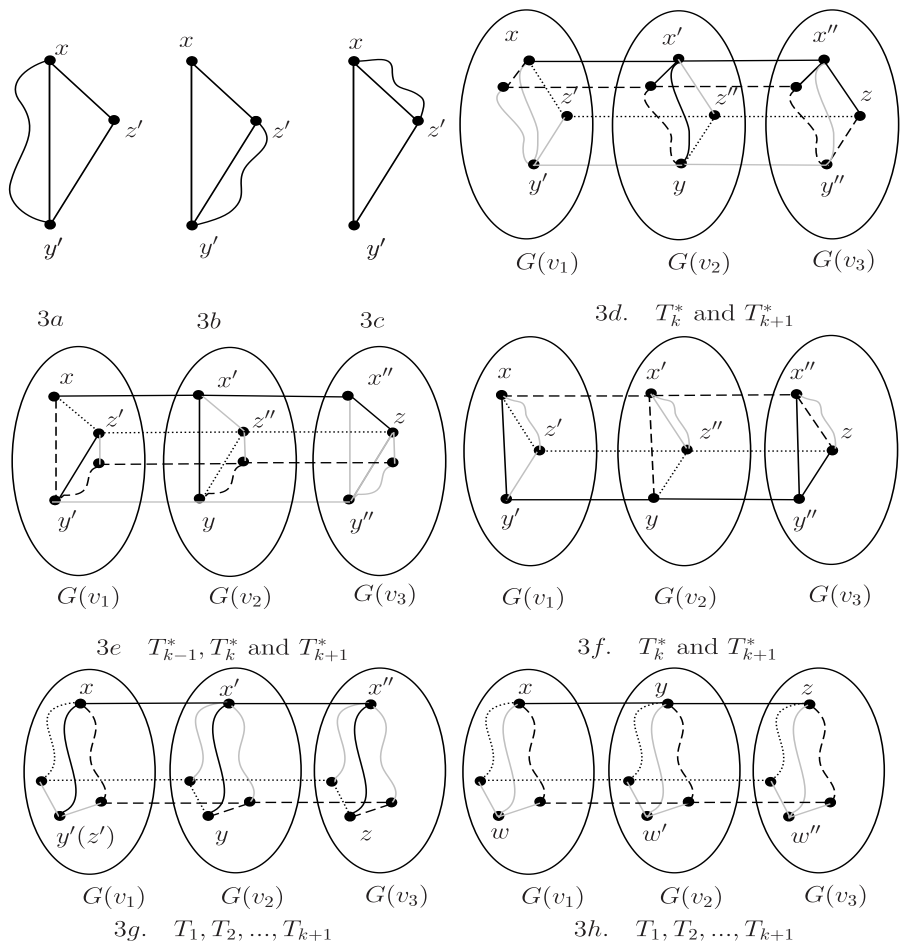

Figure 3. The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in T_i^*.

We have the following observation by the argument in the proof of Theorem 3.1.

Observation 3.1. *The $k+1$ internally disjoint S-trees consist of three kinds of edges — the edges of original trees (or paths), the edges corresponding the edges of original trees (or paths) and two-type edges.*

Note that $Q_n \cong P_2 \square P_2 \square \cdots \square P_2$, where Q_n is the n-hypercube. We have the following corollary.

Corollary 3.1. *Let Q_n be the n-hypercube with $n \geq 2$. Then $\kappa_3(Q_n) = n-1$.*

Proof. It is easy to check that $\kappa_3(Q_2) = 1$. Assume that the result holds for $\kappa_3(Q_{n-1}), n \geq 3$. By Theorem 3.1, $\kappa_3(Q_n) \geq n-1$. On the other hand, since Q_n is n-regular, $\kappa_3(Q_n) \leq n-1$ by Theorem 1.3. Thus $\kappa_3(Q_n) = n-1$. □

Example 3.1. *Let H_1 and H_2 be two complete graphs of order n, and let $V(H_1) = \{u_1, u_2, \ldots, u_n\}, V(H_2) = \{v_1, v_2, \ldots, v_n\}$. We now construct a graph G as follows:*

$$V(G) = V(H_1) \cup V(H_2) \cup \{w\}, \text{ where } w \text{ is a new vertex;}$$

$$E(G) = E(H_1) \cup E(H_2) \cup \{u_i v_j \,|\, 1 \leq i, j \leq n\} \cup \{w u_i \,|\, 1 \leq i \leq n\}$$

.

It is easy to check that $\kappa_3(G \square K_2) = \kappa_3(G) = n$ by Theorems 1.2 and 1.5.

Remark 3.1. *We know that the bounds of (i) and (ii) in Theorem 3.1 are sharp by Example 3.1 and Corollary 3.1.*

4 The Cartesian product of a connected graph and a tree

Theorem 4.1. *Let G be a connected graph and T be a tree. We have the following result.*

(i) If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G \square T) \geq \kappa_3(G)$. Moreover, the bound is sharp.

(ii) If $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G \square T) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp.

We shall prove Theorem 3.1 by a series of lemmas. Since the proofs of (i) and (ii) are similar, we only show (ii). It suffices to show that for any $S = \{x, y, z\} \subseteq G \square H$, there exist $k+1$ internally disjoint S-trees. Set $\kappa_3(G) = k$, $V(G) = \{u_1, u_2, \ldots, u_n\}$, and $V(T) = \{v_1, v_2, \ldots, v_m\}$.

Let $x \in V(G(v_i)), y \in V(G(v_j)), z \in V(G(v_k))$ be three distinct vertices. If there exists a path in T containing v_i, v_j and v_k, then we are done from Theorem 3.1. If i, j and k are not distinct integers, such a path must exist. Thus, suppose that i, j and k are distinct integers, and that there exists no path containing v_i, v_j and v_k. By Observation 2.1, there exists a tree T in H such that $d_T(v_i) = d_T(v_j) = d_T(v_k) = 1$ and all the vertices of $V(T) \setminus \{v_i, v_j, v_k\}$ have degree 2 except for one vertex, say v_4 with $d_T(v_4) = 3$. Without loss of generality, we set $i = 1, j = 2, k = 3$, $S' = \{x', y', z'\}$, where x', y' and z' are the vertices corresponding to x, y and z in $G(v_4)$, respectively. Furthermore, we assume $v_iv_4 \in E(T)$, where $1 \leq i \leq 3$. In the following argument, we can see that this assumption has no influence on the correctness of our proof. We proceed our proof by the following three lemmas.

Lemma 4.1. *If x', y' and z' are three distinct vertices, then there exist $k+1$ internally disjoint S-trees.*

Proof. Let $T_1, T_2, \ldots T_k$ be k internally disjoint S'-trees in $G(v_4)$ such that $|\{T_i \mid E(T_i) \cap E(G(v_4)[S']) \neq \emptyset\}|$ is as small as possible. We can assume $E(T_i) \cap E(G(v_4)[S']) = \emptyset$ for $1 \leq i \leq k-2$ by Lemma 2.1.

Case 1. $E(T_i) \cap E(G(v_4)[S']) = \emptyset$ for each i, where $1 \leq i \leq k$

Assume that $d_{T_i}(x') = d_{T_i}(y') = d_{T_i}(z') = 1$ for $1 \leq i \leq k_1$ and set $max\{d_{T_i}(x'), d_{T_i}(y'), d_{T_i}(z')\} = 2$ for $k_1 + 1 \leq i \leq k$. If $k_1 = k$, for $1 \leq i \leq k-1$, assume that x', y' and z' have neighbors x_i, y_i, z_i in T_i, respectively. (There maybe exist the same vertex in $\{x_i, y_i, z_i\}$). Let $T_i{}^*$ be the tree obtained from T_i by adding edges $x_ix_i', x_i'x, y_iy_i', y_i'y, z_iz_i', z_i'z$ and deleting x', y', z', where $1 \leq i \leq k-1$ and $x_i' \in V(G(v_1)), y_i' \in V(G(v_2)), z_i' \in V(G(v_3))$ are the vertices corresponding to x_i, y_i, z_i. Moreover, $T_k^* \cup T_{k+1}^*$ are shown as in Figure $4a$. Clearly, $T_1^*, T_2^*, \ldots, T_{k+1}^*$ are $k+1$ internally disjoint S-trees.

Suppose $k_1 \leq k-1$. For $1 \leq i \leq k_1$, we construct T_i^* the same as above. For T_i with $k_1 + 1 \leq i \leq k$, without loss of generality, assume that $d_{T_i}(z') = 2, N_{T_i}(z') = \{z_{i,1}, z_{i,2}\}, N_{T_i}(x') = \{x_i\}, N_{T_i}(y') = \{y_i\}$. Let $T_i{}^*$ be the tree obtained from T_i by adding edges $x_ix_i', x_i'x, y_iy_i', y_i'y, z_{i,1}z_{i,1}', z_{i,1}'z, z_{i,2}z_{i,2}', z_{i,2}'z$ and deleting $x', y'z'$, where $k_1 + 1 \leq i \leq k-1$ and $z_{i,1}', z_{i,2}' \in V(G(v_3)), x_i' \in V(G(v_1)), y_i' \in V(G(v_2))$ are the vertices corresponding to $z_{i,1}, z_{i,2}, x_i, y_i$, respectively. Moreover, T_k^* and T_{k+1}^* are as shown in Figure $4b$. Clearly, $T_1^*, T_2^*, \ldots, T_{k+1}^*$ are $k+1$ internally disjoint trees connecting $\{x, y, z\}$.

Case 2. There exists some T_i such that $E(T_i) \cap E(G(v_4)[S']) \neq \emptyset$.

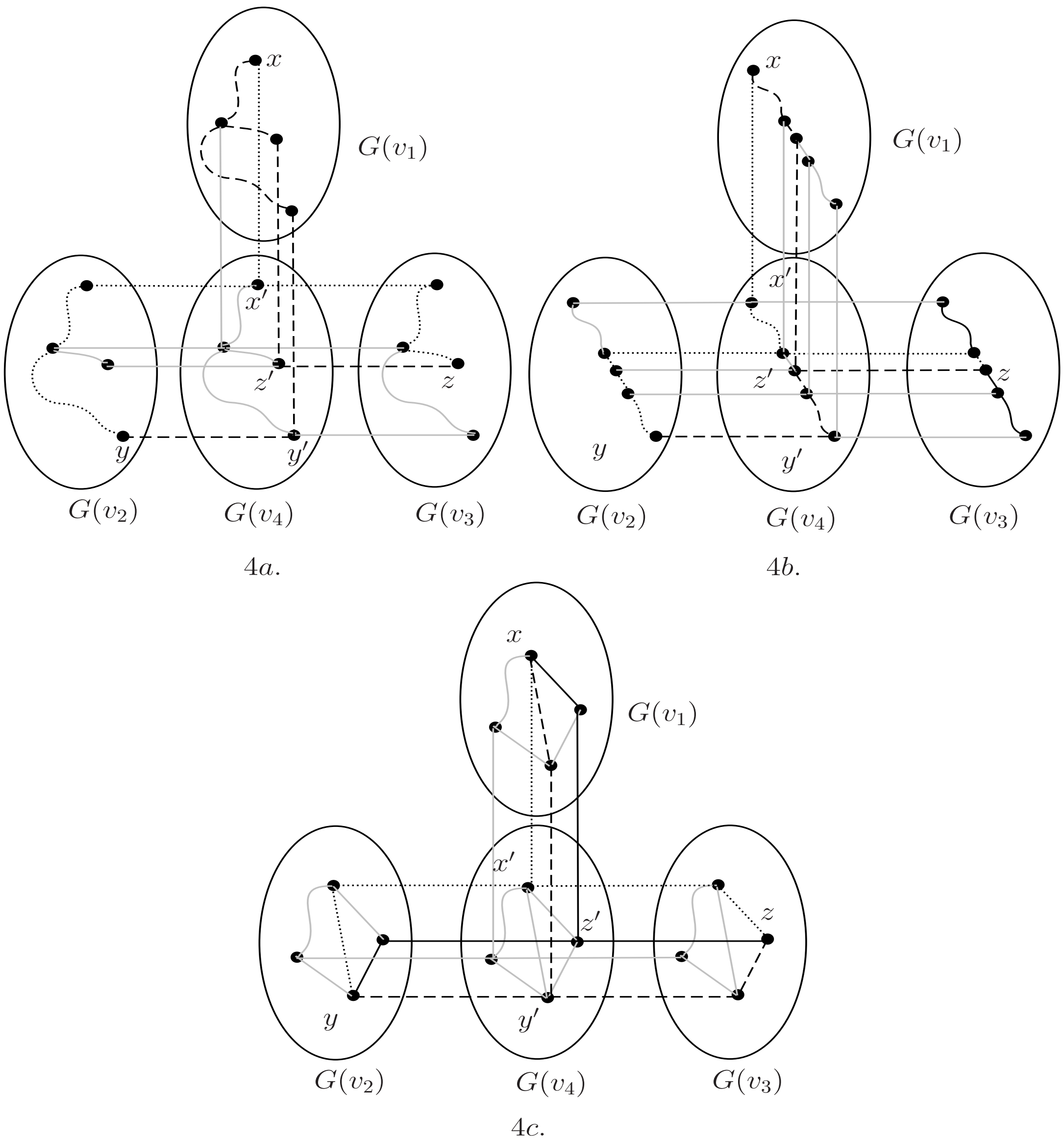

Figure 4. The edges (or paths) of a tree are shown by the same type of lines. The lightest lines stand for edges (or paths) not contained in T_i^*.

For a tree T_i such that $E(T_i) \cap E(G(v_4)[S']) = \emptyset$, we can construct T_i^* similar to that in Case 1.1.

If $E(T_{k-1}) \cap E(G(v_4)[S']) = \emptyset$ and $E(T_k) \cap E(G(v_4)[S']) \neq \emptyset$, T_k^* and T_{k+1}^* can be constructed similar to those in Figure 4a. or 4b. (depending on whether $d_{T_k}(x') = d_{T_k}(y') = d_{T_k}(z') = 1$ or not).

If $E(T_{k-1}) \cap E(G(v_4)[S']) \neq \emptyset$ and $E(T_k) \cap E(G(v_4)[S']) \neq \emptyset$, then $T_{k-1} \cup T_k$ must have the structures as shown in Figure 1e. Without loss of generality, we assume that T_k is isomorphic to P_3 which has endpoints x' and y', and the only internal vertex z'. We can obtain trees T_{k-1}^*, T_k^* and T_{k+1}^* as shown in Figure 4c. □

Lemma 4.2. *If two of x', y', z' are the same vertex in $G(v_4)$, then there exist $k+1$ internally disjoint S-trees.*

Proof. Without loss of generality, assume $y' = z'$. Since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, there exist at least $k+1$ internally disjoint x'-y' *paths* $P^1, P^2, \ldots, P^{k+1}$ in $G(v_4)$.

Assume that y_i are the only neighbor of y' in T_i, and y_i' and y_i'' are the vertices corresponding to y_i in $V(G(v_2))$ and $V(G(v_3))$, respectively, where $1 \leq i \leq k+1$.

If x' and y' are nonadjacent, let T_i be a tree obtained from P^i by adding $y_iy_i', y_i'y$, $y_i'y_i'', y_i''z$ and deleting y'; If x' and y' are adjacent, let T_i be a tree obtained from P^i by adding $y_iy_i', y_i'y, y_i'y_i'', y_i''z$. Since G is a simple graph, there exists at most one path P^i such that x and y' are adjacent on P^i. Thus, $T_1, T_2, \ldots, T_{k+1}$ are $k+1$ internally disjoint S-trees. □

Lemma 4.3. *If x', y', z' are the same vertex in $G(v_4)$, then there exist $k+1$ internally disjoint S-trees.*

Proof. Pick $w \in V(G(v_4))$ such that x', w are distinct vertices in $G(v_4)$. Since $\kappa(G) > \kappa_3(G) = k$, by Menger's Theorem, there exist at least $k+1$ internally disjoint x'-w *paths* $P^1, P^2, \ldots, P^{k+1}$. Let T_i be a tree obtained from P^i by adding $x_ix_i', x_i'y, x_i'x_i'', x_i''z$ and deleting y', where $1 \leq i \leq k+1$, x_i is the only neighbor of x in P^i, and x_i' and x_i'' are the vertices corresponding to x_i in $V(G(v_2))$ and $V(G(v_3))$, respectively. Clearly, $T_1, T_2, \ldots, T_{k+1}$ are $k+1$ internally disjoint S-trees. □

Remark 4.1. *We know that the bounds of* (*i*) *and* (*ii*) *in Theorem* 3.1 *are sharp by Example* 3.1 *and Corollary* 3.1.

Observation 4.1. *The $k+1$ internally disjoint S-trees consist of three kinds of edges — the edges of original trees (or paths), the edges corresponding the edges of original trees (or paths) and two-type edges.*

5 The Cartesian product of two general graphs

Observation 5.1. *Let G and H be two connected graphs, x, y, z be three distinct vertices in H, and $T_1, T_2, \ldots, T_k$ be k internally disjoint $\{x, y, z\}$-trees in H. Then $G\Box\bigcup_{i=1}^{k} T_i = \bigcup_{i=1}^{k}(G\Box T_i)$ has the structure as shown in Figure 5. Moreover, $(G\Box T_i) \cap (G\Box T_j) = G(x) \cup G(y) \cup G(z)$ for $i \neq j$. In order to show the structure of $G\Box\bigcup_{i=1}^{k} T_i$ clearly, we take k copies of $G(y)$, and k copies of $G(z)$. Note that, these k copes of $G(y)$ (resp. $G(z)$) represent the same graph.*

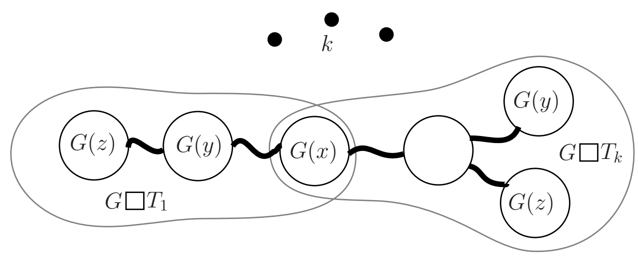

Figure 5. The structure of $G\Box\bigcup_{i=1}^{k} T_i$.

Example 5.1. Let H be the complete graph with order 4. The structure of $G\Box(T_1\cup T_2)$ are shown in Figure 6.

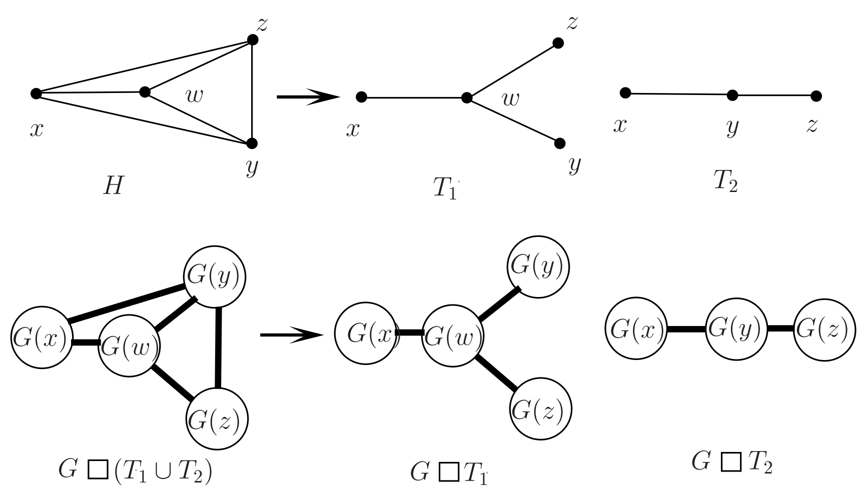

Figure 6. The structure of $G\Box(T_1 \cup T_2)$.

Theorem 5.1. *Let G and H be connected graphs such that $\kappa_3(G) \geq \kappa_3(H)$.*

(i) If $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G\Box H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.

(ii) If $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G\Box H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp.

Proof. Since the proofs of (i) and (ii) are similar, we only show (ii). Without loss of generality, we set $\kappa_3(G) := k, \kappa_3(H) := \ell$. It suffices to show that for any $S = \{x, y, z\} \subseteq G\Box H$, there exist $k + \ell$ internally disjoint S-trees. Assume $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(T) = \{v_1, v_2, \ldots, v_m\}$.

Let $x \in V(G(v_i)), y \in V(G(v_j)), z \in V(G(v_k))$ be three distinct vertices in $G\Box H$. We consider the following three cases.

Case 1. i, j, k are distinct integers.

Without loss of generality, set $i = 1, j = 2, k = 3$. Since $\kappa_3(H) = \ell$, there exist ℓ internally disjoint $\{v_1, v_2, v_3\}$-trees T_i, $1 \leq i \leq \ell$, in H. We use G_i to denote $G\Box T_i$. By Observation 5.1, we know that $G\Box \bigcup_{i=1}^{\ell} T_i = \bigcup_{i=1}^{\ell} G_i$ and $G_i \cap G_j = G(v_1) \cup G(v_2) \cup G(v_3)$ for $i \neq j$. Let y', z' be the vertices corresponding to y, z in $G(v_1)$, respectively. Consider the following three subcases.

Case 1.1. x, y', z' are distinct vertices in $G(v_1)$.

Since $\kappa_3(G(v_1)) = k$, there exist k internally disjoint $\{x, y', z'\}$-trees T'_j, $1 \leq j \leq k$, in $G(v_1)$. Let $k_0, k_1, \ldots, k_\ell$ be integers such that $0 = k_0 < k_1 < \cdots < k_\ell = k$. Similar to the proofs of Theorems 3.1 and 4.1, we can construct $k_i - k_{i-1} + 1$ internally disjoint S-trees $T_{i,j_i}, 1 \leq j_i \leq k_i - k_{i-1} + 1$, in $(\bigcup_{j=k_{i-1}+1}^{k_i} T'_j)\Box T_i$ for each i, where $1 \leq i \leq \ell$. By Observation 4.1, T_{i,j_i} and T_{r,j_r} are internally disjoint for different integers i, j. Thus $T_{i,j_i}, 1 \leq i \leq \ell, 1 \leq j_i \leq k_i - k_{i-1} + 1$ are $k + \ell$ internally disjoint S-trees.

Subcase 1.2 (exact two of x, y', z' are the same vertex in $G(v_i)$) and Subcase 1.2 (all of x, y', z' are the same vertex in $G(v_i)$) can be proved similarly, and the details are omitted.

Furthermore, Case 2 (exact two of i, j, k are the same integer) and Case 3 ($i = j = k$) can be proved similarly, and the details are also omitted.

We now show that the bound of (i) is sharp. Let K_n be a complete graph with n vertices, and P_m be a path with m vertices, where $m \geq 3$. We have $P_m = 1$, and $K_n = n - 2$ by Theorem 1.2. It is easy to check that $K_n\Box P_m = n - 2 + 1 = n - 1$ by Theorem 1.3. For (ii), Example 3.1 is a sharp example. $\square$

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